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ON THE PERIODIC SOLUTIONS OF THE PROBLEM OF THE ONE-DIMENSIONAL UNSTEADY MOTION OF A FLUID IN A TUBE†

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The periodic motion of a fluid in a tube of finite length and circular cross-section is considered. The tube has a unit at one end which changes the fluid flow rate. This unit is separated from the tube by a chamber which serves to control the flow rate and to dampen pressure oscillations. The problem of under what conditions such periodic motions are possible and what is their form and dependence on the period and durations of the first and second stages of the oscillation of the parameters determining the motion is considered.

THE THREE problems of the unsteady motion of a fluid (including periodic motion) in a system of the above type have been solved previously [1] using propositions developed by Charnyi [2]. In this case, the boundary conditions for all these problems were such that each periodic oscillation consists of two parts with durations T_1 and $T_2 = T - T_1$, where T is the period of the motion. Depending on which equation, the wave equation, the telegraph equation or the heat conduction equation, describes the motion, the three problems are referred to in [1] as the first, second and third problem, respectively.
The periodic solutions of these three problems are considered in greater detail below.

1. Let $U(z, \tau)$ be the mean velocity of the fluid in the tube, z the coordinate along the axis of the tube, and τ the time (all of these quantities are dimensionless).

The first and second problems reduce to solving the wave $(\nu = 0)$ and telegraph $(\nu > 0)$ equations

$$\frac{\partial^2 U}{\partial \tau^2} + 2\nu \frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial z^2} \tag{1.1}$$

respectively, in the domain $0 < z < 1$ with the boundary conditions

$$\frac{\partial U(0,\tau)}{\partial z} = 0, \quad b\frac{\partial U(1,\tau)}{\partial z} + U(1,\tau) = Q[U(1,\tau),\tau] \tag{1.2}$$

and without initial conditions.

Here, $Q(U, \tau)$ is either a specified periodic function of τ of the form

$$Q = Q(\tau) = \begin{cases} Q_1 & (0 < \tau < T_1) \\ Q_2 & (T_1 < \tau < T) \end{cases} \tag{1.3}$$

or this dependence on U is of the relay type

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$$Q = Q(U) = \begin{cases} Q_1 & (U < U_*) \\ Q_2 & (U > U_{**}) \end{cases} \tag{1.4}$$

Moreover, $U_* > U_{**}$ so that there is a segment of non-uniqueness $U_{**} < U < U_*$.

The quantities Q_1, Q_2, T_1, T, U_*, U_{**} are specified constant numbers. We assume that $Q_1 > Q_2$, $U_* \geqslant U_{**}$, T_1 is the duration of the first stage of the periodic motion and that $T = T_1 + T_2$ is the period of the motion (T_2 is the duration of the second stage of the oscillation). The dimensionless parameters $\nu \geqslant 0$ and $b \geqslant 0$ also occur in the conditions of the problem: ν depends on the flow conditions and the properties of the tube while b characterizes the type of chamber. When there is no chamber, $b = 0$ [1].

The third problem reduces to finding the solution of the heat conduction equation

$$2\nu\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial z^2} \quad \left(\nu > \frac{\pi}{2}\right) \tag{1.5}$$

in the domain $0 < z < 1$ without initial conditions but with boundary conditions (1.2), where $Q[U(1, \tau), \tau]$ is again determined either by formula (1.3) or (1.4).

Hence, three problems have been formulated in dimensionless variables, and the solution of each of these depends on the variables z and τ and on the dimensionless parameters b, ν, Q_1, Q_2, T_1 and T if condition (1.3) is satisfied and b, ν, Q_1, Q_2, U_*, U_{**} if (1.4) holds.

The second of conditions (1.2) (in the case of the unit when $z = 1$) is obtained from the balance of the flow rates of the fluid into and out of the chamber [2]. It is clear that, if $b = 0$, relationship (1.4) has no meaning. As will be shown below, the equality $U_* = U_{**}$ is always satisfied in the case of the first problem as a consequence of which (1.4) also has no meaning for this problem. Consequently, condition (1.4) can only hold in the case of the second and third problems when there is a chamber present ($\nu > 0$, $b > 0$).

After these remarks, we will initially dwell on the solution of the three problems with the periodic condition (1.2), (1.3) and write out the solution of the second problem, that is, (1.1)–(1.3)

$$\begin{gathered} U(z,\tau) = \begin{cases} U_1(z,\tau) & (0 \leqslant \tau \leqslant T_1) \\ U_2(z,\tau) & (T_1 \leqslant \tau \leqslant T) \end{cases} \\ U_1(z,\tau) = Q_1 + \exp(-\nu\tau)\Sigma(C_{1k}\cos\omega_k\tau + C_{2k}\sin\omega_k\tau)\cos\mu_k z, \\ U_2(z,\tau) = Q_2 + \exp[-\nu(\tau - T_1)]\Sigma[D_{1k}\cos\omega_k(\tau - T_1) + D_{2k}\sin\omega_k(\tau - T_1)]\cos\mu_k z \end{gathered} \tag{1.6}$$

Henceforth summation with respect to k is carried out from 0 to ∞, and μ_k $(k = 1, 2, \ldots)$ are the successive positive roots of the equation

$$\operatorname{ctg}\mu = b\mu \tag{1.7}$$

$\omega_k = \sqrt{(\mu_k^2 - \nu^2)}$ (if $\mu_k < \nu$, then, in (1.6), $\cos\omega_k x$ and $\sin\omega_k x$ are, respectively, replaced by $\operatorname{ch}\sigma_k x$ and $\operatorname{sh}\sigma_k x$, where $\sigma_k = \sqrt{(\nu^2 - \mu_k^2)}$; if $\mu_k = \nu$, the corresponding terms of the sums in (1.6) a replaced by $C_{1k} + C_{2k}\tau$ and $D_{1k} + D_{2k}(\tau - T_1)$).

Under the assumption that $\nu < \mu_1$, the constants C_{ik}, D_{ik} $(k = 1, 2, \ldots)$ are defined by the formulae

$$\begin{gathered} C_{jk} = \frac{a_k}{\Delta_k(T)}\{-\xi_{jk} + \alpha_{jk}^+(T_2)\exp(-\nu T_2) + \\ + \alpha_{jk}^-(T)\exp(-\nu T) - \alpha_{jk}^-(T_1)\exp[-\nu(T + T_2)]\} \quad (j = 1,2) \\ D_{1k} = a_k + \exp(-\nu T_1)(C_{1k}\cos\omega_k T_1 + C_{2k}\sin\omega_k T_1) \\ D_{2k} = (\nu/\omega_k)a_k + \exp(-\nu T_1)(-C_{1k}\sin\omega_k T_1 + C_{2k}\cos\omega_k T_1), \end{gathered} \tag{1.8}$$

$$a_k = (Q_1 - Q_2)b_k$$

$$b_k = 2\sin\mu_k / \mu_k \quad (b=0)$$

$$b_k = 2\cos\mu_k / [b\mu_k^2(1+\sin 2\mu_k / 2\mu_k)] \quad (b>0)$$

$$\Delta_k(x) = 1 - 2\exp(-\nu x)\cos\omega_k x + \exp(-2\nu x)$$

$$\xi_{1k} = 1, \quad \xi_{2k} = \nu / \omega_k$$

$$\alpha_{1k}^{\pm}(x) = \cos\omega_k x \pm (\nu / \omega_k)\sin\omega_k x$$

$$\alpha_{2k}^{\pm}(x) = \mp\sin\omega_k x + (\nu / \omega_k)\cos\omega_k x$$

The coefficients C_{jk}, D_{jk} $(j=1, 2)$ for the first problem are obtained from (1.8) by passing to the limit as $\nu \to 0$ and have been written out in [1].

The solution of the third problem, that is, problem (1.5), (1.2) and (1.3) has the form

$$\begin{aligned} U_1(z,\tau) &= Q_1 - \sum a_k\left(\frac{1-\kappa_k}{1-\rho_k\kappa_k}\right)\exp(-\lambda_k^2\tau)\cos\mu_k z \\ U_2(z,\tau) &= Q_2 + \sum a_k\left(\frac{1-\rho_k}{1-\rho_k\kappa_k}\right)\exp[-\lambda_k^2(\tau - T_1)]\cos\mu_k z \\ (\lambda_k &= \mu_k / \sqrt{2\nu}, \quad \rho_k = \exp(-\lambda_k^2 T_1), \quad \chi_k = \exp(-\lambda_k^2 T_2)) \end{aligned} \tag{1.9}$$

Here the notation of (1.6)–(1.8) has also been used.

It is clear from (1.6)–(1.9) that the series occurring in formulae (1.6) and (1.9) only converge if the period T (for the first problem) is such that there is no resonance, that is, $T \neq 2\pi m/\mu_s$ $(m, s = 1, 2, \ldots)$ [1].

Hence, under this assumption, the three problems mentioned above have a solution which is reasonable by virtue of their linearity.

Let us now consider the solution of the second and third problems with a boundary condition of the relay type, that is, problems (1.1), (1.2), (1.4) and (1.5), (1.2), (1.4), where $\nu = 0$, $b > 0$.

It can be seen that the solution of these problems is described by the same formulae (1.6)–(1.9) where T_1 and $T_2 = T - T_1$ are a priori unknown and are defined as the least roots of the equations [1]

$$U_1(1, T_1) = U_*, U_2(1, T_1 + T_2) = U_{**} \tag{1.10}$$

Let us now introduce the functions $\Phi(T_1, T_2, \nu, b)$ and $\Psi(T_1, T_2, \nu, b)$ according to the formulae

$$\begin{aligned} \Phi &= \frac{Q_1 - U_1(1,T_1)}{Q_1 - Q_2} = \frac{Q_1 - U_*}{Q_1 - Q_2} \\ \Psi &= \frac{U_2(1, T_1 + T_2) - Q_2}{Q_1 - Q_2} = \frac{U_{**} - Q_2}{Q_1 - Q_2} \end{aligned} \tag{1.11}$$

Using (1.6), (1.9) and (1.11), let us write out the expressions of the function Φ for the second and third problems, respectively

$$\begin{aligned} \Phi = \exp(-\nu T_1)\sum\frac{B_k}{\Delta_k}\{\alpha_{1k}^{+}(T_1) - \exp(-\nu T_2)\alpha_{1k}^{+}(T) - \\ -\exp(-\nu T)\alpha_{1k}^{-}(T_2) + \exp[-\nu(T + T_2)]\} \end{aligned} \tag{1.12}$$

$(B_k = b_k \cos\mu_k)$

$$\Phi = \sum \frac{B_k \rho_k (1-\chi_k)}{1-\rho_k \chi_k} \tag{1.13}$$

We will now consider formulae (1.12) and (1.13) simultaneously.

On writing out the analogous formulae for $\Psi(T_1, T_2, \nu, b)$ it can be shown that

$$\Psi(T_1, T_2, \nu, b) = \Phi(T_2, T_1, \nu, b) \tag{1.14}$$

It therefore suffices to confine ourselves to the treatment of the function Φ in the domain $T_1 \geqslant 0$, $T_2 \geqslant 0$.

At the point $T_1 = T_2 = 0$, the function $\Phi(T_1, T_2)$ is undefined and it is not continuous at this point. In this case

$$\lim_{T_1 \to 0,\, T_2 \to 0} \Phi(T_1, T_2) = \frac{T_2}{T_1 + T_2} \tag{1.15}$$

Next, it is clear that $\Phi(0, T_2) = 1$, $\Phi(T_1, 0) = 0$.

On introducing the notation

$$h(T_1) = \lim_{T_2 \to \infty} \Phi(T_1, T_2)$$

we obtain the relations

$$h(T_1) = \sum B_k \rho_k \tag{1.16}$$

$$h(T_1) = \exp(-\nu T_1) g(T_1), \quad g(T_1) = \sum B_k \alpha_{1k}^{+}(T_1) \tag{1.17}$$

for the third and second problems, respectively.

It follows from (1.16) and (1.17) that $h(0) = 1$, $\lim h(T_1) = 0$ as $T_1 \to \infty$.

Let us now consider the third problem. It follows from (1.16) that the inequality $0 < h(T_1) < 1$, $h'(T_1) < 0$ is satisfied for all $T_1 > 0$. It is clear from (1.13) that, for each value $0 < C < 1$ such that $\Phi(T_1, T_2) = C$, $T_2'(T_1) \geqslant 0$ and, when $T_2 \to \infty$, each of the curves $h(T_1)$ has a vertical asymptote $T_1 = h^{-1}(C)$.

The general form of the families of curves $\Phi(T_1, T_2) = C$ (the solid lines) and $\Psi(T_1, T_2) = C$ (the dashed lines) are shown in Fig. 1 for certain values of the parameter C. The points of intersection of these curves, T_1 and T_2, correspond to the required roots of Eqs (1.10) and, moreover, to the unique roots. It can be shown that the inequalities $0 < \Phi < 1$, $0 < \Psi < 1$, $0 < \Phi + \Psi < 1$ are satisfied when $T_1 > 0$, $T_2 > 0$. The equivalence of the solutions of the third problem with the periodic condition (1.3) and the relay type condition (1.4) follows from this. In addition, it can be seen that, in this case, a periodic solution only exists when

$$Q_2 < U_{**} < U_* < Q_1 \tag{1.18}$$

Let us now consider the solution of the second problem. It is clear that, close to the origin of coordinates and to the coordinate axes, the behaviour of the curves Φ and Ψ in the (T_1, T_2)-plane is completely analogous to that shown in Fig. 1. In particular, as in the case of the third problem, if $\Phi + \Psi = 1$ $(U_{**} = U_*)$ or $\Phi + \Psi > 1$ $(U_{**} > U_*)$, which contradicts the assumed condition $U_* > U_{**}$, then the corresponding curves $\Phi = C_1$ and $\Psi = C_2$ only intersect at the point $T_1 = T_2 = 0$.

If $h'(T_1) < 0$ for all $T_1 > 0$, where the quantity $h(T_1)$ is given by formula (1.17), the behaviour of the Φ and Ψ curves are the same as those shown in Fig. 1, and the conclusions drawn above again hold.

Let there be a segment (segments) of values of T_1 for which $h'(T_1) > 0$. The typical form of the curves $\Phi = C$ and $\Psi = C$ and their points of intersection are shown in Fig. 2; the one-to-one correspondence between the pairs (Φ, Ψ) (and, consequently, (u_*, U_{**})) and (T_1, T_2) breaks down in this case. For example, when $\Phi = 0.4$ and $\Psi = 0.2$ in our figure, there are two points of intersection while, when solving the problem with a relay type condition (1.4), it is necessary to take the point with the least values of T_1 and T_2.

Hence, if Φ and Ψ are single-valued functions of T_1 and T_2 when $T_1 > 0$, then T_1 and T_2 are multivalued functions of Φ and Ψ in the case when there are ranges of values of T_1 for which $h'(T_1) > 0$.

If there are also ranges of values of T_1 for which $h(T_1) < 0$, then inequalities (1.18) are not satisfied.

2. We will now present two examples which relate to the second problem.

Let the parameter b be large. It then follows from Eq. (1.17) that $\mu_1^2 \approx 1/b - 1/3b^2$, $\mu_k \approx \pi(k-1)$ $(k = 2, 3, \ldots)$. By virtue of (1.8) and (1.2), formula (1.17) yields

$$g(T_1) = \left(1 - \frac{1}{3b}\right)\alpha_{11}^{+}(T_1) + \frac{2}{\pi^2 b}\sum_{m=1}^{\infty}\frac{\alpha_{1,m+1}^{+}(T_1)}{m^2} \tag{2.1}$$

1. Let $\nu \ll \mu_1$, then $\omega_1 \approx \mu_1$, and from Eq. (2.1)

$$g(T_1) \approx \left(1 - \frac{1}{3b}\right)\cos\frac{T_1}{\sqrt{b}} + \frac{y(T_1)}{b}, \quad y(T_1) = T_1^2 - 2T_1 + \frac{2}{3} \;\; (0 \leqslant T_1 \leqslant 2) \tag{2.2}$$

($y(T_1)$ is a periodic function with period 2). It is clear that there are ranges of T_1 for which $g'(t_1) > 0$, $g(T_1) < 0$ and the curves $\Phi = \text{const}$ and $\Psi = \text{const}$ are arranged roughly in the same manner as in Fig. 2.

2. Let $\nu = \mu_1$. In this case, as was noted above, the first term of the series in formula (1.6) is replaced by the term $C_{11} + C_{21}\tau$, and

$$C_{11} = \frac{a_1}{\Delta_1}\{-1 + (1 + \nu T_2)\exp(-\nu T_2) + (1 - \nu T)\exp(-\nu T) - (1 - \nu T_1)\exp[-\nu(T + T_2)]\}, \tag{2.3}$$

$$C_{21} = \frac{a_1 \nu}{\Delta_1}[-1 + \exp(-\nu T_2) + \exp(-\nu T) - \exp[-\nu(T + T_2)]] \;\; (\Delta_1 = [1 - \exp(-\nu T)]^2)$$

Assuming once again that the parameter b is large, we obtain from (1.17) and (2.3) ($y(T_1)$ is a periodic function with period 2)

$$g(T_1) \approx \left(1 - \frac{1}{3b}\right)\left(1 + \frac{T_1}{\sqrt{b}}\right) + \frac{1}{2b}\left[y(T_1) + \frac{1}{\sqrt{b}}y_1(T_1)\right]$$

$$(y_1(T_1) = T_1(T_1^2/3 - T_1 + 2/3) \;\; (0 \leqslant T_1 \leqslant 2))$$

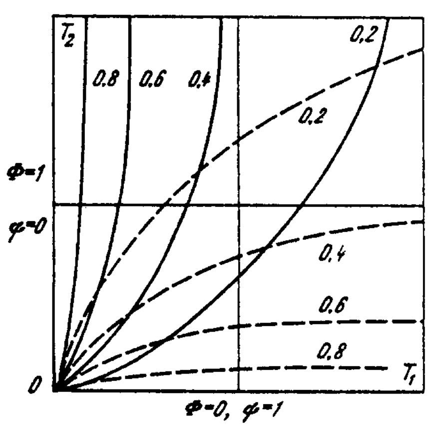

FIG. 1.

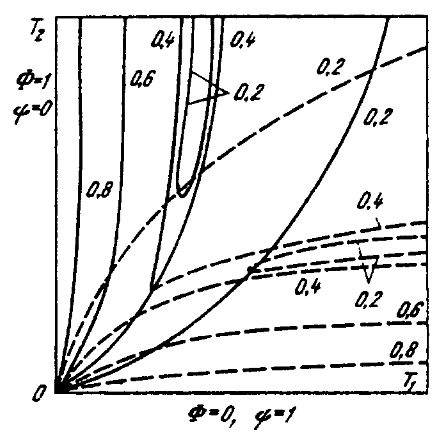

FIG. 2.

In this case, $g(T_1) > 0$, $h'(T_1) < 0$, and the picture of the distribution of the curves $\Phi = \mathrm{const}$ and $\Psi = \mathrm{const}$ in the (T_1, T_2)-plane is analogous to the picture for the third problem (Fig. 1).

Let us now turn to the periodic condition (1.2), (1.3). It is seen from Fig. 2 that, in the case of the second problem, the parameters T_1 and T_2 can also be specified in such a way that they will be the roots of Eq. (1.10) but not the least ones.

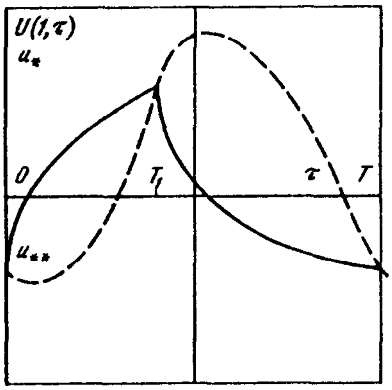

FIG. 3.

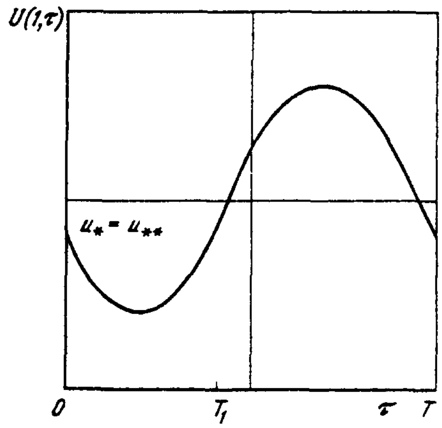

FIG. 4.

Passing to the limit as $\nu \to 0$ in formula (1.12), we can show that the equality $\Phi + \Psi = 1$, always holds in the case of the first problem, and it follows from this that $U_{**} = U_*$ and the relay type condition (1.4) is meaningless.

In concluding, we will show, on the basis of what has been described, what is the form of the oscillations for each of the three problems.

The approximate form of the periodic function $U(1, \tau)$, that is, of the mean velocity close to the unit, is shown in Figs 3 and 4 for each of the three problems: the solid line in Fig. 3 is the solution of the heat conduction equation (1.9) and the dashed line is the solution of the telegraph equation (1.6)–(1.8). The curve $U(1, \tau)$ for the wave equation (1.6)–(1.8) ($\nu = 0$) is shown in Fig. 4.

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